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An accurate hybrid block technique for second order singular problems in ordinary differential equations

Ogunniran M. O.[1*], Tijani N. A.[1], Adedokun K. A.[1], Kareem K. O.[1]

[1*1]Department of Mathematical Sciences, Osun State University, Osogbo Osun State, Nigeria.

[1]*Corresponding Author's Email: muideen.ogunniran@uniosun.edu.ng

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ABSTRACT

A block hybrid method for solution of second order singular problems in ordinary differential equations is proposed in this work. For this purpose, two basis functions were combined for the development of a continuous hybrid schemes using collocation and interpolation technique. To make the continuous scheme self-starting, a block method of discrete hybrid form was derived. The scheme was analyzed using appropriate existing definitions to investigate their stability, consistency and convergence which were then shown to be consistent, zero-stable and hence convergent. Illustrative examples have been discussed to demonstrate the validity and applicability of the technique and the results have been compared with those of existing methods.

Keywords: Orthogonal polynomial, Singular problems, Hybrid block method, Weight function, Basis function, Countinuous scheme.

2000 Mathematics subject classification: **65-02, 65L02; Secondary 65L05.**

INTRODUCTION

Some problems arising from the fields of sciences, engineering and many physical phenomena are represented by mathematical models. These mathematical models can be written in form of second order singular differential equations as:

$$P(t)u''(t) + Q(t)u'(t) + f(t,u) = g(t) \quad (1)$$

with initial conditions

$$u(0) = \alpha, u'(0) = \beta$$

where (1) is singular at $P(t) = 0$ and $f(t,u)$, $g(t)$ are non-linear continuous functions. However, solutions to certain differential equations are very difficult to obtain or cannot be found analytically due to its singularity as the approximate solutions lose their accuracy in the neighbourhood of the singular points; hence the need to derive numerical methods for the solution of such problems becomes necessary. Over the years, techniques for the numerical solution of second order singular initial value problems have been reported. These include, among others; implicit Euler method presented by Koch *et al.* (2000), Adomian decomposition method employed by Wazwaz (2001), Backward Euler method employed by Benko *et al.* (2009), Chebyshev polynomial and the collocation method reported by Changqing and Jianhua (2010), Second derivative multistep method developed by Hojjati and Parand (2011), Chebyshev wavelet finite difference method employed by Nasab *et al.* (2013), Legendre operational matrix employed by Jung *et al.* (2014). Numerous methods used in existing literature include: a new implicit method presented by Hasan *et al.* (2014), Haar wavelet collocation method employed by Shiralashetti *et al.* (2016), Chebyshev Wavelet Collocation Method (CWCM) employed by Shiralashetti and Deshi (2016), Residual-power series method present by Iryna *et al.* (2016) and non-standard finite difference schemes of Najafi and Yaghoubi (2017). Various researchers have attempted to solve these kinds of problems. Wazwaz (2001, 2005) presented series and exact solution to Lane-Emden and Emden-Fowler type of problems based on Adomian decomposition and modified Adomian decomposition methods. Koch *et al.* (2000) evaluated the approximate solution of the singular initial value problems by implicit Euler method and finally used an acceleration technique known as the Iterated defect correction to improve the approximations. Yousefi (2006) presented a numerical method for solving the Lane-Emden equations as singular initial problems using integral operator and converted Lane-Emden equation into an integral equation and then applied Legendre wavelet approximations. He exploited the Legendre wave properties together with the Gaussian integration method to reduce the integral equations to the solution of algebraic equations. Hasan and Zhu (2007, 2008) solved such a singular initial value problem by Taylor series and modified Adomian decomposition methods. Robert *et al.* (2008) presented analytic and numerical solutions for the Lane-Emden equations. They used traditional power series approach and the Homotopy Analysis Method (HAM) to obtain the solutions. A series solution obtained by HAM converged in a large interval than in the case of the corresponding traditional series solutions. Marzban *et al.* (2008) employed hybrid functions for nonlinear initial value problems with applications of Lane-Emden equations. Benko *et al.* (2009) also evaluated singular initial value problems of the Lane-Emden type equations by implicit Euler method. Hasan *et al.* (2014a, b) derived an implicit method for solving first and second singular initial value problems, which give more accurate result than those obtained by the implicit Euler and second order implicit Runge-Kutta methods. Changqing and Jianhua (2010) developed a numerical method for Lane-Emden equations using Chebyshev polynomials and the collocation method. A reliable algorithm based on Chebyshev polynomials and collocation method in order to obtain approximate solution for the Lane-Emden equation were thereby presented.

Hojjati and Parand (2011) proposed an efficient computational algorithm for solving the nonlinear Lane-Emden type equations. A new second derivative multistep methods that solve Lane-Emden type equation was thus introduced. Maria *et al.* (2013) presented a numerical solution of Emden's using Adomian polynomials. The outcome showed that Adomian's method yielded an approximate solution which overlaps with the exact solution. Jung *et al.* (2014) introduced an approach using tau method based on Legendre operational matrix differentiation for solving general form of second order linear and non linear ordinary differential equations; the actual problem is converted into a system of algebraic equations whose solutions are the Legendre coefficients. Iryna *et al.* (2016) applied residual power series method in order to obtain efficient analytical numerical solutions for a class of non linear systems of initial value problems with infinitely many singularities; the method provide analytical solution in terms of a rapid convergence series with easily computable components. Shiralashetti *et al.* (2016) defined a new method, named Haar wavelet collocation method, numerical solutions of singular initial value problem of integral equations, ordinary and partial differential equations were obtained by the method. Najafi and Yaghoubi (2017) constructed a non-standard finite difference schemes for numerical solution of non-linear Lane-Emden type equations, the use of the method and its approximations play an important role for the formation of stable numerical methods. Ogunniran *et al.* (2020) presented the solution of a class of singular problems called Lane-Emden equations using family of Runge-kutta techniques. Also, an optimized method for solving resulting stiff system which may arise from the decomposition of singular problems were presented by Ogunniran *et al.* (2020).

Orthogonal Polynomials

Orthogonal polynomials have found widespread use in all areas of science and engineering. Typically, they are used as trial functions to expand other more complicated functions in which, many at times, arise from initial or boundary value problems.

Consider the equation:

$$\int_a^b w(t)\phi_m(t)\phi_n(t)dt = h_n\sigma_{mn} \tag{2}$$

with

$$\sigma_{mn} = \begin{cases} 0, m \neq n \\ 1, m = n \end{cases}$$

where the weight function $w(t)$ is continuous and positive on $[a, b]$ such that the moments:

$$\mu = \int_a^b w(t)t^n dt, \qquad n = 0,1,2,\cdots \tag{3}$$

exist and h_n is a non-zero constant. Then the integral

$$< \phi_m, \phi_n >= \int_a^b w(t)\phi_m(t)\phi_n(t)dt$$

denotes an inner product of the polynomials ϕ_m and ϕ_n.
For orthogonality,

$$< \phi_m, \phi_n >= \int_a^b w(t)\phi_m(t)\phi_n(x)dt = 0, \qquad m \neq n.$$

The orthogonal polynomial valid in the interval $[-1,1][-1,1]$ and with respect to weight function $w(t) = t^2$ $w(t) = t^2$ was constructed (Adeniyi and Yahaya, 2014) as:

$$\begin{aligned}
\phi_0(t) &= 1 \\
\phi_1(t) &= t \\
\phi_2(t) &= \frac{1}{2}(5t^2 - 3) \\
\phi_3(t) &= \frac{1}{2}(7t^3 - 5t) \\
\phi_4(t) &= \frac{1}{8}(63t^4 - 70t^2 + 15) \\
\phi_5(t) &= \frac{1}{8}(99t^5 - 126t^3 + 35t) \\
\phi_6(t) &= \frac{1}{16}(429t^6 - 693t^4 + 315t^2 - 35) \\
\phi_7(t) &= \frac{1}{16}(715t^7 - 128t^5 + 315t^3 - 105t)
\end{aligned} \tag{4}$$

The orthogonal polynomial above is now been shifted in the interval $[0,1]$ with respect to the weight function $w(t) = t^2$. The first seven members of the shifted polynomial are given below:

$$\begin{aligned}
\phi_0^*(t) &= 1 \\
\phi_1^*(t) &= 4t - 3 \\
\phi_2^*(t) &= 15t^2 - 20t + 6 \\
\phi_3^*(t) &= 56t^3 - 105t^2 + 600t - 10 \\
\phi_4^*(t) &= 210t^4 - 504t^3 + 420t^2 - 140t + 15 \\
\phi_5^*(t) &= 729t^5 - 2310t^4 + 2520t^3 - 1260t^2 + 280t - 21 \\
\phi_6^*(t) &= 3003t^6 - 10296t^5 + 13860t^4 - 9240t^3 + 3150t^2 - 504t + 28 \\
\phi_7^*(t) &= 11440t^7 - 45045t^6 + 72072t^5 + 60060t^4 + 27720t^3 - 6930t^2 + 840t - 36.
\end{aligned}$$

Chebyshev Polynomials

The Chebyshev polynomial of the first kind is given as:

$$T_n(t) = \cos(n\arccos t) \tag{5}$$

Let $\arccos t = \theta$ implies that $t = \cos\theta$. Putting $\arccos t = \theta$ in (5), we have:

$$T_n(t) = \cos n\theta \tag{6}$$

For $n = 0n = 0$,

$$T_0(t) = \cos 0 = 1$$

and for $n = 1n = 1$,

$$T_1(t) = \cos\theta = t$$

It now implies that:

$$T_{n+1}(t) = \cos(n+1)\theta \tag{7}$$

$$T_{n-1}(t) = \cos(n-1)\theta \tag{8}$$

By identity,

$$\cos(n-1)\theta + \cos(n+1)\theta = 2\cos n\theta\cos\theta \tag{9}$$

Substituting equations (7)-(8) into (9), we have:

$$T_{n-1}(t) + T_{n+1}(t) = 2T_n(t).t \tag{10}$$

or

$$T_{n-1}(t) + T_{n+1}(t) - 2T_n(t).t = 0 \tag{11}$$

$$T_{n+1}(t) = 2t.T_n(t) - T_{n-1}(t) \tag{12}$$

The shifted chebyshev polynomials of the first kind are orthogonal on the support interval $[0,1]$ with weight function:

$$w(t) = \frac{1}{\sqrt{t - t^2}}$$

and normalized by the requirement that $T_n^*(1) = 1$.

$T_n^*(1) = 1$ satisfies the three-term recurrence relation:

$T_{n1}^*(t) = 2(2t-1)T_n^*(t) - T_{n-1}^*(t)$, for $n \geq 1$.

with starting values

$T_0^*(t) = 1, \quad T_1^*(t) = 2t - 1.$

METHODOLOGY

Introduction

We consider the IVP (1), that is

$$u''(x) = f(t, u(x), u'(x)), \quad a \leq t \leq b, \quad t_n \leq t \leq t_{n+m}$$

where m is the step number.

Let the approximate solution be given as a combination of two basis functions in the form:

$$u(t) = \sum_{r=0}^{p} a_r \phi_r^*(t) + \sum_{r=p+1}^{k} b_r T_r^*(t) \approx U(t), \quad t_n \leq t \leq t_{n+m} \tag{13}$$

where $t \in [t_n, t_{n+m}]$, $m = 1,2,3$, $a's$ and $b's$ are real coefficients to be determined, $p = u + v - 3$, $k = u + v - 1$, u is the number of collocation points, v is the number of interpolation points, $\phi_r^*(t)$ is a shifted constructed polynomial and $T_r^*(t)$ is a shifted Chebyshev polynomial, $h = t_{n+1} - t_n$ is a constant step size of the partition of interval $[a, b]$ which is given by: $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$

Differentiating (13) twice gives:

$$u''(t) = \sum_{r=0}^{p} a_r \phi_r^{*}{''}(t) + \sum_{r=p+1}^{k} b_r T^{*}{}_r{''}(t) \tag{14}$$

Derivation of the Proposed Method

Here, we consider a three-step with three off-step points method. The points are equally carefully selected to guarantee the zero-stability of the method. The three off-step points are $t_{n+\frac{3}{4}}, t_{n+\frac{7}{4}}$ and $t_{n+\frac{11}{4}}$ respectively using (13) with $p = 6, k = 8$ and $m = 3$, we have a polynomial of degree $u + v - 1$ as follows:

$$u(t) = \sum_{r=0}^{6} a_r \phi_r^{*}(t) + \sum_{r=7}^{8} b_r T_r^{*}(t) \tag{15}$$

where $t \in [t_n, t_{n+3}]$.

The second derivative is given by:

$$u''(t) = \sum_{r=0}^{6} a_r \phi^{*''}{}_r(t) + \sum_{r=7}^{8} b_r T^{*''}{}_r(t) \tag{16}$$

$$f(t, u, u') = \sum_{r=0}^{6} a_r \phi^{*''}{}_r(t) + \sum_{r=7}^{8} b_r T^{*''}{}_r(t) \tag{17}$$

Now, interpolating equation (15) at $t_{n+\frac{3}{4}}$ and $t_{n+\frac{7}{4}}$ and collocating (17) at all the points in selected interval $t_n, t_{n+\frac{3}{4}}, t_{n+1}, t_{n+\frac{7}{4}}, t_{n+2}, t_{n+\frac{11}{4}}$ and t_{n+3} gives the system of equations which were solved and the obtained unknown coefficients were substituted into (13) and (14). The resulting formulae were collocated and interpolated at the points $t_{n+\frac{3}{4}}, t_{n+\frac{7}{4}}$ and $t_{n+\frac{11}{4}}, t_{n+1}, t_{n+2}, t_{n+3}$, which leads to the block method as below:

$$\left.\begin{aligned}
&u_{n+\frac{3}{4}} - u_n - \frac{3}{4}hu'_n = h^2[\frac{5109471}{44154880}f_n - \frac{2720547}{414080}f_{n+1} - \frac{229959}{573440}f_{n+2} - \frac{16271}{573440}f_{n+3} + \frac{95321}{143360}f_{n+\frac{3}{4}} + \frac{268353}{501760}f_{n+\frac{7}{4}} + \frac{805221}{11038720}f_{n+\frac{11}{4}}]\\
&u_{n+1} - u_n - hu'_n = h^2[\frac{13723}{83160}f_n - \frac{1861}{1764}f_{n+1} - \frac{173}{280}f_{n+2} - \frac{247}{5670}f_{n+3} + \frac{3158}{2835}f_{n+\frac{3}{4}} + \frac{52}{63}f_{n+\frac{7}{4}} + \frac{302}{2695}f_{n+\frac{11}{4}}]\\
&u_{n+\frac{7}{4}} - u_n - \frac{7}{4}hu'_n = h^2[\frac{692483}{2211840}f_n - \frac{1384691}{737280}f_{n+1} - \frac{333739}{245760}f_{n+2} - \frac{617057}{6635520}f_{n+3} + \frac{4031279}{1658880}f_{n+\frac{3}{4}} + \frac{172921}{92160}f_{n+\frac{7}{4}} + \frac{14749}{61440}f_{n+\frac{11}{4}}]\\
&u_{n+2} - u_n - 2hu'_n = h^2[\frac{26468}{72765}f_n - \frac{74}{35}f_{n+1} - \frac{514}{315}f_{n+2} - \frac{314}{2835}f_{n+3} + \frac{8096}{2835}f_{n+\frac{3}{4}} + \frac{576}{245}f_{n+\frac{7}{4}} + \frac{992}{3465}f_{n+\frac{11}{4}}]\\
&u_{n+\frac{11}{4}} - u_n - \frac{11}{4}hu'_n = h^2[\frac{7962889}{15482880}f_n - \frac{33396121}{12042240}f_{n+1} - \frac{10292623}{5160960}f_{n+2} - \frac{1683715}{9289728}f_{n+3} + \frac{47685737}{11612160}f_{n+\frac{3}{4}} + \frac{775973}{215040}f_{n+\frac{7}{4}} + \frac{904475}{186336}f_{n+\frac{11}{4}}]\\
&u_{n+3} - u_n - 3hu'_n = h^2[\frac{1107}{1960}f_n - \frac{729}{245}f_{n+1} - \frac{81}{40}f_{n+2} - \frac{29}{140}f_{n+3} + \frac{158}{35}f_{n+\frac{3}{4}} + \frac{972}{245}f_{n+\frac{7}{4}} + \frac{162}{245}f_{n+\frac{11}{4}}]\\
&u'_{n+\frac{3}{4}} - u'_n = h[\frac{1089881}{5519360}f_n - \frac{784449}{501760}f_{n+1} - \frac{62793}{71680}f_{n+2} - \frac{2635}{43008}f_{n+3} + \frac{23123}{13440}f_{n+\frac{3}{4}} + \frac{18423}{15680}f_{n+\frac{7}{4}} + \frac{54441}{344960}f_{n+\frac{11}{4}}]\\
&u'_{n+1} - u'_n = h[\frac{114671}{582120}f_n - \frac{24923}{17640}f_{n+1} - \frac{433}{504}f_{n+2} - \frac{1367}{22680}f_{n+3} + \frac{5188}{2835}f_{n+\frac{3}{4}} + \frac{2536}{2205}f_{n+\frac{7}{4}} + \frac{3764}{24255}f_{n+\frac{11}{4}}]\\
&u'_{n+\frac{7}{4}} = h[\frac{607187}{3041280}f_n - \frac{86681}{92160}f_{n+1} - \frac{20923}{18432}f_{n+2} - \frac{59339}{829440}f_{n+3} + \frac{88151}{51840}f_{n+\frac{3}{4}} + \frac{1043}{576}f_{n+\frac{7}{4}} + \frac{11809}{63360}f_{n+\frac{11}{4}}]\\
&u'_{n+2} - u'_n = h[\frac{14519}{72765}f_n - \frac{2092}{2205}f_{n+1} - \frac{319}{315}f_{n+2} - \frac{40}{567}f_{n+3} + \frac{4832}{2835}f_{n+\frac{3}{4}} + \frac{4288}{2205}f_{n+\frac{7}{4}} + \frac{4448}{24255}f_{n+\frac{11}{4}}]\\
&u'_{n+\frac{11}{4}} - u'_n = h[\frac{2739209}{13547520}f_n - \frac{3603017}{4515840}f_{n+1} - \frac{57233}{645120}f_{n+2} - \frac{153065}{1161216}f_{n+3} + \frac{589633}{362880}f_{n+\frac{3}{4}} + \frac{198319}{141120}f_{n+\frac{7}{4}} + \frac{151283}{282240}f_{n+\frac{11}{4}}]\\
&u'_{n+3} - u'_n = h[\frac{4349}{21560}f_n - \frac{1611}{1960}f_{n+1} - \frac{9}{56}f_{n+2} - \frac{29}{840}f_{n+3} + \frac{172}{105}f_{n+\frac{3}{4}} + \frac{72}{49}f_{n+\frac{7}{4}} + \frac{1908}{2695}f_{n+\frac{11}{4}}]
\end{aligned}\right\} \tag{18}$$

ANALYSIS OF THE METHODS

In this section, we analyze the derived schemes by determining the order and error constant, consistency, zero stability and region of absolute stability of the schemes.

• Order and Error Constant of Main Method

Taylor Series expansion of main method of (18) gives the order $p = 7$ and error constant as 2.5677×10^{-5}.

• Order and Error Constant of the Block Method

Similarly, Taylor Series expansion on each of (18) to get the order $p = 7p = 7$ and error constants as

$$\left[\frac{84321}{1284505600}, \frac{40723}{406425600}, \frac{3553823}{1698693120}, \frac{6281}{25401600}, \frac{154067243}{416179814400}, \frac{297}{71680}, \frac{285963}{2055208960}, \frac{1861}{13547520}, \frac{172921}{1132462080}, \frac{257}{1693440}, \frac{1989845}{11098128384}, \frac{87}{501760}\right]^T$$

• Consistency

Consistency of the main method:

$\rho(z) = z^3 + \frac{5}{4}z^{\frac{3}{4}} - \frac{9}{4}z^{\frac{7}{4}}$racteristic polynomials of the main method are given by:

and

$$\sigma(z) = \frac{86969}{26492928} - \frac{31573}{258048}z^{\frac{3}{4}} + \frac{41563}{100352}z^{\frac{7}{4}} + \frac{468663}{2207749}z^{\frac{11}{4}} + \frac{323569}{802816}z + \frac{60693}{114688}z^2 - \frac{34451}{1032192}z^3$$

Consequently, (18) is consistent since it satisfies the following:

the order of the method is $p = 7 > 1$

$\alpha_0 = 1, \alpha_{\frac{3}{4}} = \frac{5}{4}, \alpha_{\frac{7}{4}} = -\frac{9}{4}$,thus: $\sum_{j=0}^{k} \alpha_j = 1 + \frac{5}{4} - \frac{9}{4} = 0$

$$\rho(1) = 1 + \frac{5}{4} - \frac{9}{4} = 0$$

$$\rho'(z) = 3z^2 + \frac{15}{16}z^{-\frac{1}{4}} - \frac{63}{16}z^{\frac{3}{4}}$$

$$\rho'(1) = 3 + \frac{15}{16} - \frac{63}{16} = 0$$

$$\therefore \rho(1) = \rho'(1) = 0$$

$$\rho''(z) = 6z - \frac{15}{64}z^{-\frac{5}{4}} - \frac{189}{64}z^{-\frac{1}{4}}$$

$$\rho''(z) = \frac{45}{16}$$

$$\sigma(1) = \frac{86969}{26492928} - \frac{31573}{258048} + \frac{41563}{100352} + \frac{468663}{2207744} + \frac{323569}{802816} + \frac{60693}{114688} - \frac{34451}{1032192} = \frac{45}{32}$$

$$2!\,\sigma(1) = 2\left(\frac{45}{32}\right) = \frac{45}{16}$$

$$\Rightarrow \rho''(1) = 2!\,\sigma(1)$$

Zero-Stability of the Method

Zero-stability of Main Method

The first and second characteristics polynomial of (18) are given as:

$$\rho(z) = z^3 + \frac{5}{4}z^{\frac{3}{4}} - \frac{9}{4}z^{\frac{7}{4}} \qquad (19)$$

equating (19) to zero and solving for z gives:

$z = 0$

$z = 1$

Zero-Stability of Block Method

By definition,

$$\rho(z) = det[z\bar{A} - \bar{B}] \qquad (20)$$

$$\rho(z) = det\left[z\begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix} - \begin{bmatrix} 0&1&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix} \right]$$

$$\rho(z) = det\begin{bmatrix} z & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & z & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & z & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & z & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z \end{bmatrix}$$

$\rho(z) = z^{11}(z-1)$

Equating to zero and solving for z gives:

$z^{11}(z-1) = 0$

$\Rightarrow z = 0,0,0,0,0,0,0,0,0,0,0,1$

Hence,the method is zero-stable.

Convergence

The block method (18) is convergent since it satisfies the necessary and sufficient conditions of consistency and zero-stability.

Region of Absolute Stability

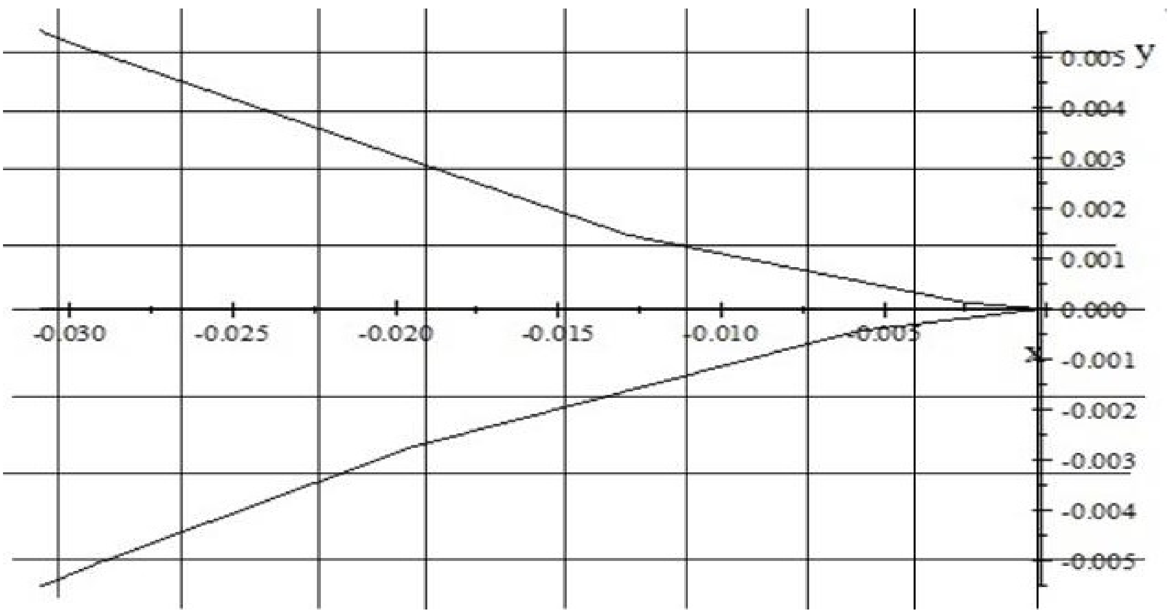

Figure 1: Region of Absolute Stability of Method (18)

NUMERICAL RESULTS

In this section, we shall consider the application of the derived schemes to three test problems for efficiency and accuracy of the methods. Error contained here is Absolute which was obtained as the absolute difference between the derived method and Exact. CBH$M_{3,3}$ $M_{3,3}$ is the Three-step Block Hybrid Method of three off-step points.

Problem 1: Linear non-homogeneous problem

$$u'' + cot(\frac{t}{2})u' - 1 = 0, 0 \le t \le 1$$
$$u(0) = 1, u'(0) = 0, h = 0.05$$

$$u(t) = 3 - t\frac{cos(\frac{t}{2})}{sin(\frac{t}{2})}$$

Problem 2: Linear singular non-homogeneous Lane-Emdem equation

$$u'' + \frac{8}{t}u' + tu = t^5 - t^4 + 44t^2 - 30t, 0 \le t \le 1$$
$$u(0) = 0, u'(0) = 0, h = \frac{1}{32}$$

Exact solution:

$$u(t) = t^4 - t^3$$

Problem 3: Linear singular initial value problem

$$u'' + \frac{2}{t}u' = 2(2t^2 + 3)u, 0 \le t \le 1$$
$$u(0) = 1, u'(0) = 0, h = \frac{1}{32}$$

Exact solution:

$$u(t) = exp(t^2)$$

Table 1: Numerical Comparison for Problem 1.

x	Exact	CBHM $_{3,3}$	Error in CBHM $_{3,3}$	Error in Benko *et al.* (2008)
0.1	1.001666944	1.001804713	1.37796e-4	1.56289e-4
0.2	1.006671115	1.006819294	1.48179e-4	1.56353e-4
0.3	1.015022549	1.015174206	1.51657e-4	1.56462e-4
0.4	1.026738050	1.026891456	1.53406e-4	1.56613e-4
0.5	1.041841318	1.041995780	1.54462e-4	1.56805e-4
0.6	1.060363114	1.060518286	1.55172e-4	1.57039e-4
0.7	1.082341489	1.082497173	1.55684e-4	1.57313e-4
0.8	1.107822064	1.107978137	1.56073e-4	1.57626e-4
0.9	1.136858375	1.137014754	1.56379e-4	1.57977e-4
1.0	1.169512278	1.169668106	1.56628e-4	1.58363e-4

Table 2: Numerical Comparison for Problem 2.

x	Exact	CBHM $_{3,3}$	Error in CBHM $_{2,3}$	Error in Shiralashetti
				et al. (2016)
0.03125	-0.00002956390380	-0.00002956390387	7.0000e-14	1.9000e-5
0.09375	-0.00074672698980	-0.00074672698980	0.0000	1.0600e-4
0.15625	-0.00321865081800	-0.00321865081800	0.0000	1.1400e-4
0.21875	-0.00817775726600	-0.00817775726300	3.0000e-12	1.4300e-4
0.28125	-0.01599025726000	-0.01599025726000	0.0000e	1.5800e-4
0.34375	-0.02665615082000	-0.02665615082000	0.0000	1.5900e-4
0.40625	-0.03980922699000	-0.03980922699000	0.0000	1.4900e-4
0.46875	-0.05471706392000	-0.05471706391000	1.0000e-11	1.2700e-4
0.53125	-0.07028102873000	-0.07028102876000	3.0000e-11	9.3000e-5
0.59375	-0.08503627779000	-0.08503627778000	1.0000e-11	4.6000e-5
0.65625	-0.09715175628000	-0.09715175629000	1.0000e-11	1.2000e-5
0.71875	-0.10443019870000	-0.10443019870000	0.0000	8.1000e-5
0.78125	-0.10430812840000	-0.10430812840000	0.0000	1.6200e-4
0.84375	-0.09385585784000	-0.09385585794000	1.0000e-10	2.5500e-4
0.90625	-0.06977748871000	-0.06977748881000	1.0000e-10	3.5900e-4
0.96875	-0.02841091156000	-0.02841091167000	1.1000e-10	4.7500e-4

Table 3: Numerical Comparison for Problem 3.

x	Exact	CBHM $_{3,3}$	Error in CBHM $_{3,3}$	Error in Shiralashetti
				and Deshi (2016)
0.03125	1.000977039	1.001250479	2.7344e-4	7.0955e-4
0.09375	1.008827800	1.009169430	3.4163e-4	7.1675e-4
0.15625	1.024714526	1.025074458	3.5993e-4	7.2766e-4
0.21875	1.049014931	1.049388850	3.7391e-4	7.4561e-4
0.28125	1.082314239	1.082702986	3.8874e-4	7.6845e-4
0.34375	1.125428737	1.125834795	4.0605e-4	7.9887e-4
0.40625	1.179439190	1.179865951	4.2676e-4	8.3819e-4
0.46875	1.245736053	1.246187654	4.5160e-4	8.8406e-4
0.53125	1.326079126	1.326560468	4.8134e-4	3.9594e-3
0.59375	1.422675228	1.423192090	5.1686e-4	4.6815e-3
0.65625	1.538278698	1.538837904	5.5920e-4	5.3890e-3
0.71875	1.676321086	1.676930738	6.0965e-4	6.1228e-3
0.78125	1.841078539	1.841748313	6.6977e-4	6.9174e-3
0.84375	2.037888173	2.038629703	7.4153e-4	7.8063e-3
0.90625	2.273428541	2.274255900	8.2735e-4	8.8252e-3
0.96875	2.556084416	2.557014737	9.3032e-4	1.0013e-2

CONCLUSION

A block hybrid method has been proposed for solving second order singular ordinary differential equations via the interpolation and collocation approach. The developed method is zero-stable, consistent and convergent. The numerical results show that the methods perform favourably well when compared with existing methods.

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