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On Fuglede – Putnam Theorems for Moore-Penrose Invertible Operators on Hilbert Spaces

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Abstract

The Fuglede-Putnam Theorems were first established by Fuglede [1950] and Putnam [1951] where they gave results on bounded operators commuting with normal operators. Fuglede [1950] showed that if $P \in B(H)$ commutes with a normal operator T, then it also commutes with its adjoint T^*. Putnam[1951] extended the results to two normal operators, T and S commuting with a bounded operator $P \in B(H)$. The study on commuting operators plays a vital role in operator theory where quantities are represented by operators and if operators are commuting, then it means the quantities can be observed at the same time. In this paper, it is established that Fuglede-Putnam theorems holds true for EP operators under certain commutativity conditions involving the operators TT^*, T^*T^+, SS^* and S^*S^+ as well as for injective operators or operators with dense range. Results on Fuglede-Putnam type commutativity theorems when the adjoint is replaced with Moore-Penrose inverse and T being EP operator is replaced by either injective operator or an operator with dense range are also shown.

Mathematics subject classification: 47B20

Keywords: Moore-Penrose inverse, EP operator, Fuglede-Putnam Theorems

Introduction

For operators with closed range there exist a unique inverse often referred to as the Moore-Penrose inverse. For $T \in C(H)$, with $R(T) = R(T^*)$, then T is an EP operator. The Fuglede-Putnam Theorems do not hold in general for EP operators but under some conditions they hold true. Fuglede [1950] came up with a result generally referred to as Fuglede theorem which was later generalized by Putnam [1951]. Fuglede [1950] showed that for $P \in B(H)$ and normal operator T such that [P,T]=0, then $[P, T^*] = 0$. Putnam [1951] extended the same result to two normal operators. In particular, the author showed that if T, S are normal operators and P ∈ B(H) a bounded linear operator such that $C[S, T]P = 0$, then $C[S^*, T^*]P = 0$. Mahmood [2017] extended the same results to two normal operators and two bounded operators. Mecheri [2004] came up with results involving Fuglede - Putnam theorems on p-hyponormal operators while Bachir and Prasad [2020] extended the theorems to $(\sigma, \phi) -$ normal operator. Several other scholars have worked on Fuglede-Putnam theorems such as Duggal [2001]. A simple extension of theorems was given by Gong [1987] as well as Gupta and Patel [1988]. A characterization of operator $T \in C(H)$ where $R(T) = R(T^*)$ was discussed by Brock [1990]. In particular, the author showed that if an operator has $R(T) = R(T^*)$, then $[T, T^+] = 0$. It is known that if $[T, T^*] = 0$ t and R(T) is closed, then T is an EP-operator.

Several authors have worked on operators as well as matrices whose range equal to the range of its adjoint such as Djordjevic and Koliha [2007], Campbell and Meyer [1975], Katz and Pearl [1966], Meyer [1970], Mosic [2017] among others. Johnson et al. [2021] extended the results of Fuglede [1950] and Putnam [1951] to operators whose ranges equal to the ranges of their adjoint operators. In particular, they showed Fuglede and Putnam results holds for operators whose range (null space) equal to the range (null space) of their adjoint under some conditions. The authors did that by replacing T^* by T^+. The study of Fuglede-Putnam theorem on EP operators and injective linear operators (and operators with dense range) is not fully exhausted. This paper makes a continuation of this study by giving results under conditions different from the ones of other scholars. This paper shows that the Fuglede and Putnam results holds for operators whose ranges equal to the ranges of their adjoint operators (EP operators) under conditions different from the ones stated by Johnson et al. [2021]. That is, it is shown among others that, if $T \in B_C(H)$ has range equal to the range of its adjoint and $P \in B(H)$ where $[P,T] = 0$ and $[P, T^*T^+] = 0$, then $[P, T^*] = 0$. A number of corollaries pertaining to these theorems are deduced and extend them to two EP operators as well as two bounded operators. It also establishes that the Fuglede-Putnam results hold for either injective linear operators and linear operators with dense range satisfying some given conditions. Moreover, corollaries on Fuglede-Putnam type commutativity theorems of Johnson et al. [2021] are deduced. Particularly, Corollaries 3.2, Corollary 3.3 and Corollary 3.4.

Notations and Terminology

In this paper, B(H) represents space of bounded linear operators on H, $B_C(H)$ −subset of B(H) with closed range, L(H)- space of linear operators on H, R(T)-range of T and $\overline{R(T)}$- closure of R(T). N(T) - kernel of T and $N(T)^\perp$ orthogonal complement of N(T). If T and S are two operators, then we will use $[T,S] = TS - ST = 0$ to imply that the operators are commuting and $C[S,T]P = 0$ for $PT = SP$. For $T \in B_C(H)$, T^* is the adjoint of T where $< Tx, y > = < x, T^*y >$. The unique operator X X satisfying the conditions:(a) $TXT = T$, (b) $XTX = X$, (c) $(TX)^* = TX$, (d) $(XT)^* = XT$ is often referred to as Moore-Penrose inverse of T.

Given T ∈ B(H), then it's:

- Normal if $[T, T^*] = 0$.
- p-hyponormal if $(T^*T)^p \geq (TT^*)^p$ for $0 < p \leq 1$.
- Orthogonal projection if $T^2 = T = T^*$.
- (σ, ϕ) − normal operator $(0 \leq \sigma \leq 1 \leq \phi)$ if $\sigma^2 T^*T \leq TT^* \leq \phi^2 T^*T$.

An operator or a matrix whose range equal to the range of its adjoint or their null spaces are equal is referred to as an EP-operator.

Results

Remark 3.1

In Theorem A and Theorem B below, Johnson et al. [2021] gave results on a bounded operator commuting with Moore-Penrose inverse of an EP operator. In Corollary 3.2, 3.3 and 3.4 below, commutativity of $P \in B(H)$, T^+T (TT^+) and S^+S (SS^+) where T and S are EP-operators.

Theorem A (Johnson et al. [2021])

Let $P \in B(H)$ and T, S have ranges equal to the ranges of their adjoints.

(i) If [P,T]=0, then $[P, T^+] = 0$.

(ii) If $PT = SP$, then $C[S^+, T^+]P$.

Theorem B (Johnson et al. [2021])

Let $P, Q \in B(H)$ and T, S be EP operators. If:

(i) $PT = TQ$ and $QT = TP$, then $PT^+ = T^+Q$ and $QT^+ = T^+P$.

(ii) $PT = SQ$ and $QT = SP$ then $PT^+ = S^+Q$ and $QT^+ = S^+P$.

Corollary 3.2

Let $P \in B(H)$ and T be an EP operator. If $[P,T] = 0$, then

(i) $[P, T^+T] = 0$.

(ii) $PTT^+ = TT^+P$.
(iii) $T^+TP = PTT^+$.

Proof

(i) If $P \in B(H)$ and T be an EP operator such that $[P, T] = 0$, then by Theorem A (i) above

$PT^+ = T^+P$. Thus $PT^+T = T^+PT = T^+TP$. Hence, $[P, T^+T] = 0$.

(ii) If $PT = TP$, then , $[P, T^+] = 0$. Hence, $TT^+P = TPT^+ = PTT^+$. Thus, $[P, TT^+] = 0$.

(iii) If T is an EP-operator, then A(i) above indicates that $[P, T^+] = 0$, so that

Note that for a bounded operator T in H, $R(T) \oplus N(T^*)$ but if $R(T) = R(T^*)$ then $R(T) \oplus N(T)$.

Corollary 3.3

Let $P \in B(H)$ and T, S be EP operators where $PT = SP$, then $C[S^+S, T^+T]P = 0$ and $C[SS^+, TT^+]P = 0$.

Proof

Since $P \in B(H)$ and T,S are EP operators such that $C[S, T]P = 0$, then Theorem A (ii) above, $C[S^+, T^+]P = 0$. Thus $PT^+T = S^+PT = S^+SP$. Hence, $C[S^+S, T^+T]P = 0$
Also, $PTT^+ = SPT^+ = SS^+P$. Thus, $PTT^+ = SS^+P$ implying $C[SS^+, TT^+]P = 0$.

Corollary 3.4

Suppose $P, Q \in B(H)$ and T, S are EP- operators where $PT = SQ$ and $QT = SP$ then,
(i) $S^+SP = PT^+T$ and $SS^+P = PTT^+$.
(ii) $S^+SQ = QT^+T$ and $SS^+Q = QTT^+$.

Proof

(i) If T, S be an EP operators such that $PT = SQ$ and $QT = SP$, then by Theorem B (ii) above $PT^+ = S^+Q$ and $QT^+ = S^+P$. Thus, $PT^+T = S^+QT = S^+SP$. Hence, $PT^+T = S^+SP$. Also, $PTT^+ = SQT^+ = SS^+P$. Thus, $PTT^+ = SS^+P$. Hence, $C[S^+S, T^+T]P = 0$ and $C[SS^+, TT^+]P = 0$.
(ii) Similarly, $QT^+T = S^+PT = S^+SQ$ implying $QT^+T = S^+SQ$. Also, $QTT^+ = SPT^+ = SS^+Q$. Hence $QTT^+ = SS^+Q$. Hence, $C[S^+S, T^+T]Q = 0$ and $C[SS^+, TT^+]Q = 0$.

In the sequel, proofs are given for Fuglede-Putnam Theorems on EP operator under conditions different from the ones of Johnson et al. [2021] in Theorem C and Theorem D below and results extended to two bounded operators and two EP operators.

Theorem C (Johnson et al. [2021])

Let T be an EP- operator and $P \in B(H)$ where [P,T]=0 and $T^*TP = PT^*T$, then $[P, T^*] = 0$.

Theorem D (Johnson et al. [2021])

Suppose $T, S \in C(H)$ are EP-operators and $P \in B(H)$ where $TP = PT$, then $[T^*, P]$ =0 in each these cases:
(i) $C[S^*S, T^*T]P = 0$.
(ii) $C[S^+S^*, T^+T^*]P = 0$.

Theorem 3.5

Let $T \in C(H)$ be an EP-operator and $P \in B(H)$. If $[P, T] = 0$ and $[TT^*, P] = 0$, then $[P, T^*] = 0$.

Proof

Since, $R(T) = R(T^*)$ and $P \in B(H)$ such that PT=TP, then $PT^+ = T^+P$. . So that

$PT^* = P(TT^+T)^* = P(T^+T)^*T^* = PT^+TT^* = T^+PTT^*$. If $[P, TT^*] = 0$, then
$PT^* = T^+PTT^* = T^+TT^*P = (T^+T)^*T^*P = (TT^+T)^*P = T^*P$. Thus $PT^* = T^*P$.

Example 3.6
Define an EP-operator T on H by
$T(x_1, x_2, x_3, x_4, \dots) = (x_1 - x_2, x_1 + x_3, 2x_1 - x_2 + x_3, x_4, \dots)$ and $P \in B(H)$ defined by
$P(x_1, x_2, x_3, \dots) = (x_2, -x_1 + x_2 - x_3, -2x_1 + x_2, x_4, \dots)$ so that
$T^*(x_1, x_2, x_3, x_4, \dots) = (x_1 + x_2 + 2x_3, -x_1 - x_2, x_2 + x_3, x_4, \dots)$. so that
$PTT^*(x_1, x_2, x_3, x_4, \dots) = (x_1 + 2x_2 + 3x_3, -4x_1 - 2x_2 - 6x_3, -3x_1 - 3x_3, x_4, \dots)$.
$TT^*P(x_1, x_2, x_3, x_4, \dots) = (-7x_1 + 6x_2 - x_3, -8x_1 + 6x_2 - 2x_3, -15x_1 + 12x_2 - 3x_3, x_4, \dots)$. This implies $PTT^* \neq TT^*P$.
By computing PT^* and T^*P,
we have $PT^*(x_1, x_2, x_3, x_4, \dots) = (-x_1 - x_3, -2x_1 - 2x_2 - 4x_3, -3x_1 - 2x_2 - 5x_3, , x_4, \dots) \neq T^*P$
$(x_1, x_2, x_3, x_4, \dots) = (-5x_1 + 4x_2 - x_3, 2x_1 - 2x_2, -3x_1 + 2x_2 - x_3, x_4, \dots)$.

This implies that the conditions cannot be overlooked for the conclusion to hold.

Corollary 3.7
Suppose $T \in C(H)$ is an EP-operator and $P, Q \in B(H)$ such that $PT = TQ$ and $QT = TP$. If:
(i) $C[Q, P]TT^* = 0$, then $[P, T^*] = 0$.
(ii) $C[P, Q]TT^* = 0$, then $[Q, T^*] = 0$.
Proof
(i) If $PT = TQ$ and $QT = TP$, then $PT^+ = T^+Q$ so that
$PT^* = P(TT^+T)^* = P(T^+T)^*T^* = PT^+TT^* = T^+QTT^*$. If $QTT^* = TT^*P$, then
$PT^* = T^+QTT^* = T^+TT^*P = T^*P$. Thus $PT^* = T^*P$.
(ii) Tracing (i) above the proof carries through. That is interchanging the roles of P and Q.

Theorem 3.8
Suppose $T, S \in C(H)$ are EP-operators and $P, Q \in B(H)$ such that $C[P, Q]T = 0$ and $C[Q, P]T = 0$. If:
(i) $QTT^* = TT^*Q$, then $C[P, Q]T^* = 0$.
(ii) $PTT^* = TT^*P$ then $PT^* = T^*Q$.
Proof
(i) Since $PT = TQ$ and $QT = TP = 0$, then $PT^+ = T^+Q$. So that
$PT^* = P(TT^+T)^* = P(T^+T)^*T^* = PT^+TT^* = T^+QTT^*$. If $[Q, TT^*] = 0$, then
$PT^* = T^+QTT^* = T^+TT^*Q = T^*Q$. Thus $PT^* = T^*Q$ implying $C[P, Q]T^* = 0$.
(ii) Similarly, from the stated conditions $C[Q, P]T^+ = 0$. Thus,
then
$QT^* = T^+PTT^* = T^+TT^*P = T^*PQT^* = T^+PTT^* = T^+TT^*P = T^*P$. Hence, $[Q, P]T^* = 0$.
Remark 3.9
From the literature, an operator that has closed range and commutes with its adjoint is an EP operator but the converse is not true. As a result, in the Theorem 3.10 below, it is shown that Fuglede and Putnam results holds for operators whose null spaces equal to those of their adjoint operators (EP operators) under the given conditions and extend it to two bounded operators in Theorem 3.12 that follow.
Theorem 3.10
Suppose $T, S \in C(H)$ are EP-operators and $P \in B(H)$ such that $PT = TQ$. If $C[SS^*, TT^*]P = 0$, then $C[S^*, T^*]P = 0$.

Proof
If T, S are EP-operators and $P \in B(H)$ such that PT=TQ, then $C[S^+, T^+]P = 0$.
So that $PT^* = P(TT^+T)^* = P(T^+T)^*T^* = PT^+TT^* = S^+PTT^*$. If $C[SS^*, TT^*]P = 0$, then
$PT^* = S^+PTT^* = S^+SS^*P = S^*P$. Thus $C[S^*, T^*]P = 0$.

Example 3.11
Define EP-operators T and S on H by
$T(x_1, x_2, x_3, x_4, \dots) = (x_1 + x_3, 0, x_3, x_4, \dots)$, $S(x_1, x_2, x_3, x_4, \dots) = (x_1 + x_2, x_2, 0, x_4, \dots)$ and
$P \in B(H)$ defined by $P(x_1, x_2, x_3, \dots) = (x_1 - x_3, x_3, 2x_2, x_4, \dots)$. In this case,

$PT(x_1,x_2,x_3,x_4,\dots) = (x_1,x_3,0,x_4,\dots)$ and $SP(x_1,x_2,x_3,x_4,\dots) = (x_1,x_3,0,x_4,\dots)$. Thus $PT = SP$.
Also, $T^*(x_1,x_2,x_3,x_4,\dots) = (x_1,0,x_1+x_3,x_4,\dots)$. and $S^*(x_1,x_2,x_3,x_4,\dots) = (x_1,x_1+x_2,0,x_4,\dots)$.
So that $PTT^*(x_1,x_2,x_3,x_4,\dots) = (x_1,x_1+x_3,0,x_4,\dots)$ and
$SS^*P(x_1,x_2,x_3,x_4,\dots) = (2x_1-x_3,x_1,0,x_4,\dots)$. This implies $PTT^* \neq SS^*P$. By computing PT^* and S^*P we have,
$PT^*(x_1,x_2,x_3,x_4,\dots) = (-x_3,x_1+x_3,0,x_4,\dots)$ and
$S^*S^*P(x_1,x_2,x_3,x_4,\dots) = (x_1-x_3,x_1,0,x_4,\dots)$. Thus $PT^* \neq S^*P$.
Thus the conditions are vital for the results to hold.

Theorem E (Mahmood [2017])
Suppose $P,Q \in B(H)$ and $T,S \in B(H)$ are normal operators such that:
(i) $PT = TQ$ and $QT = TP$, then $PT^* = T^*Q$ and $QT^* = T^*P$.
(ii) $PT = SQ$ and $QT = SP$, then $PT^* = S^*Q$ and $QT^* = S^*P$.

Theorem 3.12
Let $T,S \in C(H)$ be EP-operators and $P,Q \in B(H)$ such that $PT = SQ$ and $QT = SP$. If:
(i) $C[SS^*,TT^*]P = 0$, then $QT^* = S^*P$.
(ii) $C[SS^*,TT^*]Q = 0$, then $PT^* = S^*Q$.

Proof
(i) The stated conditions imply, $PT^+ = S^+Q$ and $QT^+ = S^+P$ so that,
$QT^* = Q(TT^+T)^* = Q(T^+T)^*T^* = QT^+TT^* = S^+PTT^*$. If $PTT^* = SS^*P$, then
$QT^* = S^+PTT^* = S^+SS^*P = S^*P$. Thus $QT^* = S^*P$.
(ii) Again, $PT^* = P(TT^+T)^* = P(T^+T)^*T^* = PT^+TT^* = S^+QTT^*$. If $QTT^* = SS^*Q$, then
$PT^* = S^+QTT^* = S^+SS^*Q = S^*Q$. Thus $PT^* = S^*Q$.

Theorem 3.13
Suppose $T,S \in C(H)$ are EP-operators and $P,Q \in B(H)$ such that $PT = SQ$ and $QT = SP$.
If $QTT^* = SS^*P$, then $C[S^*,T^*]P = 0$.

Proof
The stated conditions imply $PT^+ = S^+Q$. So that,
$PT^* = P(TT^+T)^* = P(T^+T)^*T^* = PT^+TT^* = S^+QTT^*$. If $QTT^* = SS^*P$, then
$PT^* = S^+QTT^* = S^+SS^*P = S^*P$. Thus $C[S^*,T^*]P = 0$

Theorem 3.14

Let $T \in C(H)$ be an EP-operator and $P \in B(H)$ such that $PT = TP$. Given $[P,T^*T^+] = 0$, then $[P,T^*] = 0$.

Proof

We first note that $T^* = (TT^+T)^* = (T^+T)^*T^* = T^+TT^*$. Also, $PT^* = PT^+TT^*$. Now,

$[P,T^+] = 0$. Thus, $PT^* = PT^+TT^* = T^+PTT^*$. Now,

$PT^* = T^+PTT^* = T^+TPT^* = T^+TPT^*TT^+ = T^+TPT^*T^+T$. Since $[P,T^*T^+] = 0$, then

$PT^* = T^+TPT^*T^+T = T^+TT^*T^+PT = T^+TT^*T^+TP = T^+T(TT^+T)^*P = T^+TT^*P = T^*P$.

Thus, $[P,T^*] = 0$.

It is important to note that $T^* = T^+TT^* = T^*TT^+$. Since $T^* = (TT^+T)^* = (T^+T)^*T^* = T^+TT^*$ and
$T^* = (TT^+T)^* = T^*(TT^+)^* = T^*TT^+$.

Theorem 3.15

Suppose $T,S \in C(H)$ are EP-operators and $P,Q \in B(H)$ such that $PT = SQ$ and $QT = SP$.

If $C[S^*S^+, T^*T^+]P = 0$, then $PT^* = S^*Q$.

Proof

The assumed conditions indicates that $PT^+ = S^+Q$ and as a result,
$PT^* = PT^+TT^* = S^+QTT^* = S^+SPT^* = S^+SPT^*TT^+ = S^+SPT^*T^+T$. Since, $C[S^*S^+, T^*T^+]P = 0$, then
$PT^* = S^+SPT^*T^+T = S^+SS^*S^+PT = S^+SS^*S^+SQ = S^+SS^*Q = S^*Q$. Thus, $PT^* = S^*Q$.

Theorem 3.16

Suppose $T, S \in C(H)$ are EP-operators and $P \in L(H)$ where $PT = SP$. If $C[S^*S^+, T^*T^+]P = 0$, then $PT^* = S^*P$.

Proof

The assumed conditions indicates that $PT^+ = S^+P$.
Thus, $PT^* = P(T^+T)^*T^* = PT^+TT^* = S^+PTT^* = S^+SPT^* = S^+SPT^*TT^+ = S^+SPT^*T^+T$.
By assumption $C[S^*S^+, T^*T^+]P = 0$. Hence,
$PT^* = S^+SPT^*T^+T = S^+SS^*S^+PT = S^+SS^*S^+SP = S^+SS^*P = S^*P$.
Thus, $PT^* = S^*P$.

Corollary 3.17

Suppose $T, S \in C(H)$ are EP-operators and $P, Q \in B(H)$ such that $PT = SQ$ and $QT = SP$.
If $PT^*T^+ = S^*S^+Q$, then $C[S^*, T^*]P = 0$.

Proof

From the stated conditions, $PT^+ = S^+Q$ and consequently follows that,
$PT^* = PT^+TT^* = S^+QTT^* = S^+SPT^* = S^+SPT^*TT^+ = S^+SPT^*T^+T$. Since $PT^*T^+ = S^*S^+Q$, then
$PT^* = S^+SPT^*T^+T = S^+SS^*S^+QT = S^+SS^*S^+SP = S^+SS^*P = S^*P$. Thus, $PT^* = S^*P$.

Remark 3.18

It is worth noting that interchanging the roles of P and Q in Theorem 3.16 above, we have $QT^* = S^*P$. Again, in Corollary 3.17 note that interchanging the roles of Q and P then $C[S^*, T^*]Q = 0$. Theorem 3.19 below extends Theorem C and Theorem D above to injective linear operators as well as linear operators with dense range without necessarily the operators being EP operators.

Theorem 3.19

Suppose $P, T \in C(H)$ and PT=TP , then $PT^* = T^*P$ under each of these conditions:

(i) $PT^*T = T^*TP$ and $\overline{R(T)} = H$.

(ii) $[P, TT^*] = 0$ and T is one − to − one.

Proof

(i) If $PT = TP$, then $T^*PT = T^*TP = PT^*T$. That is $PT^*T = T^*PT$. This implies that $PT^*T - T^*PT = 0$ and $(PT^* - T^*P)T = 0$. Since T has a dense range, then $PT^* - T^*P = 0$. Hence, $[P, T^*] = 0$.

(ii) If $[P, TT^*] = 0$ and $[P, T] = 0$, then $TPT^* = PTT^* = TT^*P$. So that $TT^*P - TPT^* = 0$ and $T(T^*P - PT^*) = 0$. Since $N(T) = 0$, then $T^*P - PT^* = 0$. Hence, $[P, T^*] = 0$.

Corollary 3.20

Given a quasiaffinity T and $P \in B(H)$, let PT=TP. If $[P, T^*T] = 0$ or $[P, TT^*] = 0$, then $[P, T^*] = 0$.

Proof

T being a quasiaffinity implies T is one-to-one and has dense range. Hence by Theorem 3.19 above, the results follow.

Corollary 3.21

Let $P, Q, T \in B(H)$ satisfy $PT = TQ$. If:

(i) $C[P, Q]T^*T = 0$ and T has a dense range, then $[P, T^*] = 0$.

(ii) $C[P, Q]TT^* = 0$ and T is one-one, then $[Q, T^*] = 0$.

Proof

(i) If $C[P, Q]T^*T = 0$ and $C[P, Q]T = 0$, then $PT^*T = T^*TQ = T^*PT$ implying $PT^*T - T^*PT = 0$. Thus $(PT^* - T^*P)T = 0$. If $\overline{R(T)} = H$, then $PT^* - T^*P = 0$. Implying $[P, T^*] = 0$.

(ii) By the hypothesis, if $PTT^* = TT^*Q$ and $PT = TQ$. Thus,
$TT^*Q = PTT^* = TQT^*$. So that, $TQT^* - TT^*Q = 0$. This implies $T(QT^* - T^*Q) = 0$. Since T is one-to-one, then $QT^* - T^*Q = 0$ and $QT^* = T^*Q$ implying $[Q, T^*] = 0$.

Corollary 3.22

Let $P, Q, S, T \in B(H)$ be such that $PT = SQ$. If:

(i) $PT^*T = S^*SQ$ and T has a dense range, then $C[S^*, T^*]P = 0$.

(ii) $PTT^* = SS^*Q$ and S is one – one, then $C[S^*, T^*]Q = 0$.

Proof

(i) If $PT^*T = S^*SQ$ and $PT = SQ$, then $PT^*T = S^*SQ = S^*PT$ implying $PT^*T = S^*PT$ and $PT^*T - S^*PT = 0$. Thus, $(PT^* - S^*P)T = 0$. If T has a dense range, then $PT^* - S^*P = 0$. Thus $PT^* = S^*P$ implying $C[S^*, T^*]P = 0$.

(ii) If $PTT^* = SS^*Q$ and $PT = SQ$, then $SQT^* = PTT^* = SS^*Q$ and $SQT^* = SS^*Q$. Hence, $SQT^* - SS^*Q = 0$. Thus, $S(QT^* - S^*Q) = 0$. If S is one-one, then $QT^* - S^*Q = 0$. Thus, $QT^* = S^*Q$ implying $C[S^*, T^*]Q = 0$.

Remark 3.23

By replacing the adjoint of T with T^+ in Theorem 3.19 above and Corollary 3.21 above, the following Fuglede-Putnam type commutativity theorems are obtained as illustrated in following results.

Theorem 3.24

Suppose $P, T \in C(H)$ and PT=TP , then $PT^+ = T^+P$ in each of these conditions:

(i) $PT^+T = TT^+P$ and $\overline{R(T)} = H$.

(ii) $[P, TT^+] = 0$ and T is one-one.

Proof

Follows from Theorem 3.19 above by replacing the adjoint of T with its Moore-Penrose inverse. That is substitute T^* for T^+ in Theorem 3.19 above.

Corollary 3.25

Let $P, Q, T \in B(H)$ be such that $PT = TQ$. If

(i) $PT^+T = T^+TQ$ and $\overline{R(T)} = H$, then $[P, T^+] = 0$.

(ii) $C[P, Q]TT^+ = 0$ and T is one – one, then $[Q, T^+] = 0$.

Proof

Follows from Corollary 3.21 above by replacing T^* by T^+.

Corollary 3.26

Let $P, Q, S, T \in B(H)$ where $PT = SQ$. If:

(i) $PT^+T = S^+SQ$ and T has a dense range, then $C[S^+, T^+]P = 0$.

(ii) $PTT^+ = SS^+Q$ and S is one – one, then $C[S^+, T^+]Q = 0$.

Proof

Substituting T^* for T^+ in Corollary 3.22 above the results follows.

4.0 Conclusion and Recommendations

The Fuglede-Putnam theorems holds true for EP operators and injective linear operators (and linear operators with dense range) satisfying the aforementioned conditions. Further research can be done to ascertain whether the same results (or parallel results) can be obtained under different conditions.

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