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Square-Metrically Equivalent Operators and Their Closure under Unitary Equivalence

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ABSTRACT

This paper introduces a new class of equivalence relation called square-metrically equivalent operators. It explores the properties of this class and its relation to other equivalence classes like unitary equivalence and metric equivalence. The study is justified by the need for new insights in operator theory, and it has practical relevance for simplifying complex operator structures. The results show that square-metric equivalence is closed under unitary equivalence and that metric equivalence implies square-metric equivalence. This research opens new pathways in operator theory and proposes new methods for handling operator-related problems.

Keywords: Square-metric equivalence, metric equivalence, class (Q) operators.

INTRODUCTION

In the realm of operator theory, understanding equivalence relations among operators is pivotal to grasping their inherent properties and behaviors. Traditional equivalence classes, like unitary and metric equivalence, have been extensively studied and applied, yielding significant insights. However, as the field evolves, there remains a constant need to delve into new equivalence relations that can offer a deeper understanding and novel applications. This paper introduces a new equivalence relation known as square-metrically equivalent operators. Despite the considerable research on unitary and metric equivalence by scholars such as Furuta ([1]), Jibril ([2], [3], [4]), Mahmood Kmail Shihab ([5]), and Nzimbi et al. ([6], [7]), there remains a notable gap in the literature concerning equivalence relations that capture more nuanced operator properties. Square-metric equivalence seeks to fill this gap, providing a fresh perspective and new tools for analyzing operator behavior. Previous studies have not fully explored the implications of such an equivalence class on the broader landscape of operator theory.

This research aims to answer several key questions: What are the defining properties of square-metrically equivalent operators? How does square-metric equivalence relate to unitary and metric equivalence? What are the implications of square-metric equivalence for the class of (Q) operators? Finally, how can square-metric equivalence simplify the analysis of complex operator structures? The problem addressed by this study is the need for new equivalence relations that offer deeper insights and practical tools for analysis within operator theory. Traditional equivalence classes, while useful, do not capture all the nuanced behaviors of operators. This research aims to explore how square-metric equivalence operates and its potential applications in simplifying and analyzing operator structures.

The objectives of this study are to define and characterize the class of square-metrically equivalent operators, establish the relationship between square-metric equivalence and other well-known equivalence classes such as unitary and metric equivalence, explore the implications of square-metric equivalence on the class of (Q) operators, and demonstrate the practical relevance of square-metric equivalence in simplifying complex operator structures. This research is significant because it introduces a novel concept in operator theory that bridges existing gaps and opens new avenues for theoretical and practical applications. By showing that square-metric equivalence is preserved under unitary equivalence and that metric equivalence implies square-metric equivalence, this study provides robust theoretical foundations and practical tools for mathematicians. The findings contribute to a more comprehensive understanding of operator equivalence and its applications, potentially influencing future research and methodologies in the field.

MAIN RESULTS

Two operators $S \in B(H)$ and $T \in B(H)$ are said to be square-metrically equivalent operators, denoted by $S \sim_m 2\ T$, provided $(S*)^2S^2=(T*)^2T^2$.

Theorem 1. *If* S *is a square-normal operator and* $T \in B(H)$ *is unitarily equivalent to* S*,then* T *is square- normal.*

Proof. Since $T = U^*SU$,with U being unitary and S square- normal , we have :$(T^*)^2T^2$

$= (U^*S^*U)^2(U^*SU)^2$

$= (U^*S^*SU)^2$

$= (U^*SS^*U)^2$

$= (TU^*S^*U)^2$

$= (TU^*UT^*)^2$

$= (TT^*)^2$

$= (T^*)^2T^2$

$= T^2(T^*)^2$

Which proves the claim.

Corollary 2. *An operator* $T \in B(H)$ *is square-normal if and only if* T *and* T^* *are square-metrically equivalent.*
Proof. The proof follows from Theorem 1

Theorem 3. *Two operators* S *and* T *are square- metrically equivalent provided* $(S^*)^2S^2$
and $(T^*)^2T^2$ *are unitarily equivalent.*

Proof. The proof is simple and follows from $(S^*)^2S^2 = U(T^*)^2T^2U^*$ with U as the unitaryoperator.

Theorem 4. *Square-metric equivalence is an equivalence relation.*
Proof. (i) $S \sim_m 2\ S$ since;

$(S^*)^2S^2= U (S^*)^2S^2 U^*$.

(ii) If $S \sim_m 2\ T$; it means

$$(S^*)^2S^2 = U(T^*)^2T^2U^* \quad (0.1)$$

Pre-multiplying 0.1 on both sides by U^* and post multiplying the same by U we obtain:

$$U^*(S^*)^2S^2U = U^*U(T^*)^2T^2U^*U = U^*(S^*)^2S^2U = (T^*)^2T^2 \quad (0.2)$$

hence $T \sim_m 2\ S$.

(iii) We need to illustrate that if $S \sim_m 2\ Q$ and $Q \sim_m 2\ T$, it follows that $S \sim_m 2\ T$. $(S^*)^2S^2= U (Q^*)^2Q^2 U^*$ and $(Q^*)^2Q^2= G (T^*)^2T^2 G^*$ are unitary operators. Then; $(S^*)^2S^2= U (Q^*)^2Q^2 U^* = UG (T^*)^2T^2 UG^*$ where U and G are unitary operators. Then ; $(S^*)^2S^2= U (Q^*)^2Q^2 U^* = M (T^*)^2T^2 M^*$ where $M = UG$. Hence; $S \sim_m 2\ T$.

Lemma 5. *Given operators* S *and* T *which are linear and bounded on a Hilbert space* H*, with* $S \sim_m 2\ T$, then*:*
(i) Whenever T *is isometric, then* S *is also isometric. (ii) Whenever* T *is a contraction, then* S *is also a contraction.*
Proof. (i) Proof follows from $(S^*)^2S^2 = (T^*)^2T^2 = I$.
(ii) This follows from $S^2x = T^2x \leq x$ for all $x \in H$.

Proposition 6. *If* $S \in B(H)$ *and* $T \in B(H)$ *are metrically equivalent operators, thenthey are square-metrically equivalent.*
Proof. Since S and T are metrically equivalent, then;
$(S^*)^2S^2 = S^*S^*SS = SS^*SS^* =$
$TT^*TT^* = T^*T^*TT = (T^*)^2T^2$.

The converse need not be true. We give an example of a square-metrically equivalent

Operators which are not metrically equivalent.

Example 7. $S = \begin{bmatrix} 0 & \frac{1}{2} \\ -2 & 0 \end{bmatrix}$ *and* $T = \begin{bmatrix} -i & 0 \\ i & i \end{bmatrix}$ *in* C^2.

A simple computation shows that;

$\mathcal{E}(S^*)^2S^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = (T^*)^2T^2$ *hence square-metrically equivalent, but* $S^*S = \begin{bmatrix} 4 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$ *and* $T^*T = \begin{bmatrix} -2 & -1 \\ -1 & -1 \end{bmatrix}$

hence they are not metrically equivalent.

Remark 8. *The following results illustrate the spectral picture of square-metrically equiva- lent operators where square-metrically equivalent operators do not necessarily need to have an equal spectra.*

Proposition 9. *Square-metrically equivalent operators need not have equal spectra.*

Consider square metrically equivalent operators

represented by: $S = \begin{bmatrix} 0 & 1 \\ & \end{bmatrix}$ and $T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ in C^2.

Computation illustrates that $\sigma(S) = \{-1, 1\}$ and $\sigma(T) = \{1, 1\}$. It is equivalently clear that $W(S) /= W(T)$. Hence square-metric equivalence does not preserve numerical range.

Theorem 10. *If* $S \in B(H)$ *and* $T \in B(H)$ *are unitarily equivalent, then they are square- metrically equivalent.*

Proof. We proof by contradiction. The converse of Theorem 10 is not generally true. Consider two operators in C^2 represented by the matrices

$S = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $T = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$

A simple computation shows that $(S^*)^2S^2 =$

$(T^*)^2T^2 = \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix}$ which implies S and T are square-

metrically equivalent. However8 8

$\sigma(S) = \{0, 2\} = \sigma(T) = \{0, -2\}$. This means that S and T are not similar hence cannot

be unitarily equivalent.

Theorem 11. *If* S *and* T *are square-metrically equivalent operators and* S *and* T *arequasi-isometries, then* S *and* T *are metrically equivalent.*

Proof. The proof is straight forward and follows from the theorem itself. Since $S \sim_m^2 T$, $(S^{*2})S^2 = (T^{*2})T^2$ rearranging we get, $S^{*2}S^2 = T^{*2}T^2$. Since S and T are quasi-isometries

, that is, $S^{*2}S^2 = S^*S$ and $T^{*2}T^2 = T^*T$; we have $S^*S = T^*T$

Remark 12. *The following results establishes the square- metric equivalence relation ofoperators on* Q *-operators class.*

Theorem 13. *Let* $S \in (Q)$ *and* $T \in (Q)$. *Then* $S \in (Q)$ *and* $T \in (Q)$ *are said to besquare- metrically equivalent* (Q)*-operators if and only if* S *and* T *are isometries.*

Proof. Given that $S \in (Q)$ and $T \in (Q)$, then

$S^{*2}S^2 = (S^*S)^2$ and

$T^{*2}T^2 = (T^*T)^2$. This implies;

$S^{*2}S^2 = (S^*S)^2 = (S^*)^2S^2$ and

$T^{*2}T^2 = (T^*T)^2 = (T^*)^2T^2$.

Since both S and T are isometries, the following is realized;

$$(S^*)^2S^2 = I = (S^*S)^2 \tag{0.3}$$

Similarly,

$$S^2(S^*)^2 = I \tag{0.4}$$

$$(T^*)^2T^2 = I = (T^*T)^2 \tag{0.5}$$

; equivalently

$$T^2(T^*)^2 = I \tag{0.6}$$

From 0.4 and 0.6 it is observed that; $(S^*)^2S^2 = I = (T^*)^2T^2$ and so; $(S^*)^2S^2 = (T^*)^2T^{2.}$

For the converse, Since $S \in (Q)$ and $T \in (Q)$ are isometries, we have;

$S^{*2}S^2 = (S^*S)^2 = (S^*)^2S^2 = I$ and $T^{*2}T^2 = (T^*T)^2 = (T^*)^2T^2 = I$, hence; $(S^*)^2S^2 = I$

$= (T^*)^2T^2$ and so; $(S^*)^2S^2 = (T^*)^2T^2$.

DISCUSSION AND RECOMMENDATION

Theorems 1, 3, and 4 highlight significant properties and relationships of square-metrically equivalent operators, establishing their preservation under unitary equivalence and confirming their status as an equivalence relation. Future research should explore further properties and applications, extend the concept to broader classes of operators, and investigate practical applications in various fields. Additionally, incorporating these concepts into educational curricula will equip future mathematicians with a deep understanding of modern operator theory.

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